Int. J. Solids Structures, 1977, Vol. 13, pp. 1293–1300. Pergamon Press. Printed in Great Britain

ON SMALL BENDING AND STRETCHING OF SANDWICH-TYPE SHELLS†

E. REISSNER
Department of Applied Mechanics and Engineering Sciences, University of California, San Diego, La Jolla, CA 92037, U.S.A.

(*Received* 20 *January* 1977; *revised* 13 *June* 1977)

Abstract—Recent interest in the theory of elastic Cosserat surfaces suggested reconsideration of the authors' theory of sandwich shells, as containing the essence of the difference between Cosserat-type and ordinary two-dimensional shell theory, the incorporation of transverse normal stress deformation into the equations of the two-dimensional theory, over and above the incorporation of transverse shear stress deformation. In reviewing the earlier work the possibility of certain improvements and simplifications became apparent. These, in addition to the fact that it seemed appropriate to indicate the relation of the work in[2] to Cosserat elastic surface theory, led to the writing of the present paper.

INTRODUCTION

Recent work on the subject of direct formulations of two-dimensional theories for three-dimensional problems, with emphasis on the concept of a Cosserat surface[1], suggested a reconsideration of the author's earlier work on sandwich-type shells[2].

In what follows we show that our earlier derivation of a two-dimensional sandwich-type shell theory from a suitably idealized three-dimensional formulation contains in a natural way what appears to be the essence of the difference between Cosserat-type elastic surface theory and ordinary two-dimensional shell theory, to whit, an incorporation of the effect of transverse normal stress deformation into the equations of the two-dimensional theory, over and above the incorporation of the effect of transverse shear stress deformation.

The developments which follow are in the main equivalent to our earlier developments. However, in reviewing our earlier work the possibility of certain improvements and simplifications became apparent. It is these improvements and simplifications, in addition to recognition of the fact that sandwich-type shell theory as formulated by us does in fact contain the essence of Cosserat elastic surface theory, which led to the writing of the present paper.

To indicate the nature of the following considerations it may be worthwhile to quote (with some slight modifications in wording) from the Introduction to our earlier work[2], as follows. "In this report an extension of the classical theory of small bending and stretching of thin elastic shells is considered. Instead of a homogeneous shell we consider a shell constructed in three layers: A core layer of thickness 2c with elastic constants E_c, G_c, ν_c and two face layers of thickness t with elastic constants E_t, G_t, ν_t. In the developments certain restrictive assumptions are made, which somewhat limit the applicability of the results. In so doing formulas are obtained which are as compact as possible while still describing the essential characteristics of the sandwich-type shell." (Our reconsideration shows that the earlier formulas, while being "compact", were not in fact "as compact as possible".)

"The thickness ratio t/c is assumed small compared to unity; at the same time the ratio tE_t/cE_c is assumed large compared to unity. This latter assumption means that the face material is so much stiffer than the core material that the contribution of the core layer to stress couples and tangential stress resultants is negligible. It is known that for flat plates this assumption necessitates the taking account of the effect of transverse shear deformation. The same would be expected to be true for curved shells."

"A further effect... is the effect of transverse normal stress deformation. We show that this effect arises in a manner which is typical for shells and has no counterpart in plate theory. It may be likened, roughly, to what happens in the bending of curved tubes."

†Preparation of this paper has been supported by the Office of Naval Research.

STATICS OF SANDWICH-TYPE SHELL

In order to derive a complete system of equations we first consider the statics of the face layers and of the core layer. Combination of the results obtained for these two components will lead to those equations of equilibrium which hold for elements of a shell regardless of the constructional nature of the elements, and in addition to relations which are associated with the sandwich-type nature of the elements.

Coordinate system on shell. In formulating differential equations we use a curvilinear coordinate system ξ_1, ξ_2, ζ, such that ξ_1 and ξ_2 are lines-of-curvature coordinates on the middle surface of the composite shell, and ζ the normal distance from this middle surface (Fig. 1). The linear element in this system of coordinates is of the form

$$ds^2 = \alpha_1{}^2(1 + \zeta/R_1)^2 \, d\xi_1{}^2 + \alpha_2{}^2(1 + \zeta/R_2)^2 \, d\xi_2{}^2 + d\zeta^2. \tag{1}$$

Statics of face layers. The face layers are treated as shells of thickness t, with negligible bending stiffness about their own middle surfaces. Because of this they are designated in what follows as face membranes.

The middle surfaces of the face membranes are given with reference to the three-dimensional system of coordinates by $\zeta = \pm(c + t/2) \approx \pm c$, with the linear elements on these two middle surfaces given by $ds^2 = \alpha_1{}^2(1 \pm c/R_1)^2 \, d\xi_1{}^2 + \alpha_2{}^2(1 \pm c/R_2) \, d\xi_2{}^2$.

The components of external load intensity on the upper and lower face membranes are designated by p_{iu}, q_u and p_{il}, q_l, respectively. The core layer stresses which act upon the two membranes are designated by $\tau_{i\zeta u}$, $\sigma_{\zeta u}$, $\tau_{i\zeta l}$, $\sigma_{\zeta l}$, and the stress resultants acting over the cross sections of the membranes by $N_{iju} = N_{jiu}$ and $N_{ijl} = N_{jil}$ (Fig. 2).

There are then three equations of force equilibrium for the elements of each of the two membranes. Writing $\alpha_{nu} = \alpha_n(1 + c/R_n)$, $a_{nl} = \alpha_n(1 - c/R_n)$, we have two tangential component

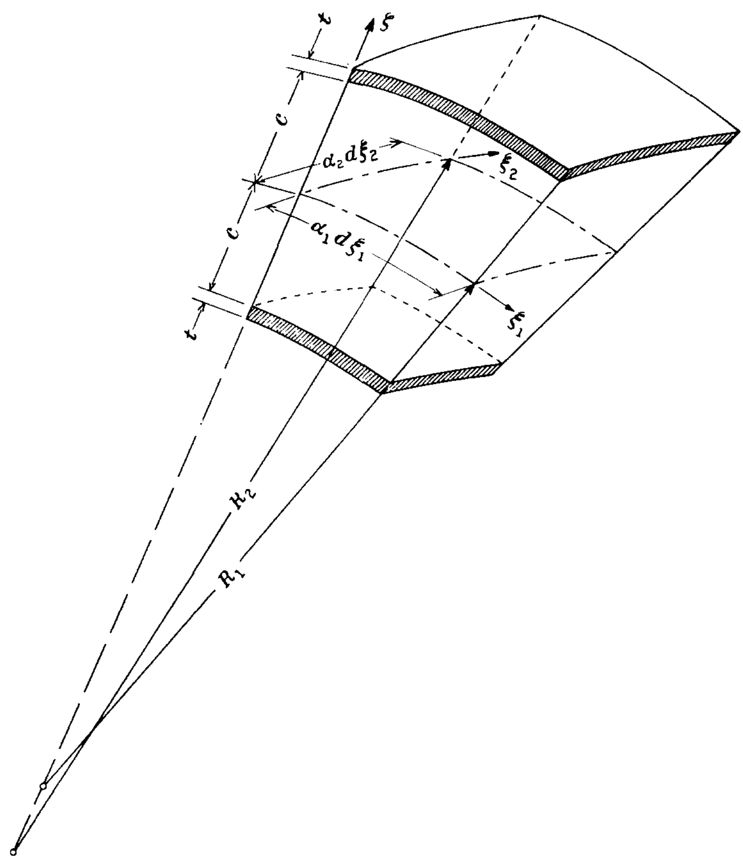

Fig. 1.

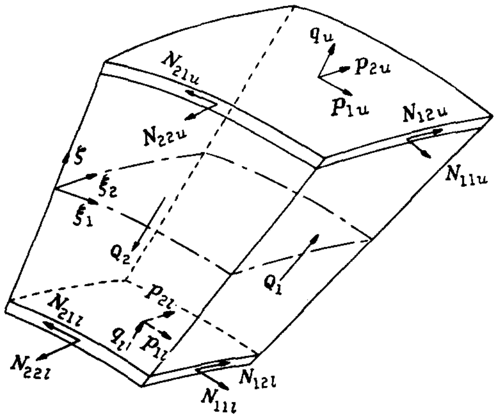

Fig. 2.

equations for forces in the direction ξ_1,

$$
\begin{aligned}
(\alpha_{2u}N_{11u})_{,1} + (\alpha_{1u}N_{21u})_{,2} + \alpha_{1u,2}N_{12u} - \alpha_{2u,1}N_{22u} + \alpha_{1u}\alpha_{2u}(p_{1u} - \tau_{1\zeta u}) &= 0, \\
(\alpha_{2l}N_{11l})_{,1} + (\alpha_{1l}N_{21l})_{,2} + \alpha_{1l,2}N_{12l} - \alpha_{2l,1}N_{22l} + \alpha_{1l}\alpha_{2l}(p_{1l} + \tau_{1\zeta l}) &= 0,
\end{aligned}
\tag{2}
$$

two analogous equations for forces in the direction ξ_2, and two equations for forces in the normal direction,

$$
\begin{aligned}
\alpha_{1u}\alpha_{2u}\left[\frac{N_{11u}}{R_1 + c} + \frac{N_{22u}}{R_2 + c} - q_u + \sigma_{\zeta u}\right] &= 0, \\
\alpha_{1l}\alpha_{2l}\left[\frac{N_{11l}}{R_1 - c} + \frac{N_{22l}}{R_2 - c} - q_l - \sigma_{\zeta l}\right] &= 0.
\end{aligned}
\tag{3}
$$

Statics of core layer. Assuming that the components of stress σ_1, σ_2, τ_{12} are negligible so that only the transverse stresses σ_ζ, and $\tau_{i\zeta}$ need to be retained we have three differential equations of equilibrium of the form

$$
\begin{gathered}
\frac{\partial}{\partial\zeta}\left[\left(1 + \frac{\zeta}{R_1}\right)^2\left(1 + \frac{\zeta}{R_2}\right)\tau_{1\zeta}\right] = 0, \qquad \frac{\partial}{\partial\zeta}\left[\left(1 + \frac{\zeta}{R_2}\right)^2\left(1 + \frac{\zeta}{R_1}\right)\tau_{2\zeta}\right] = 0, \\
\frac{\partial}{\partial\zeta}\left[\left(1 + \frac{\zeta}{R_1}\right)\left(1 + \frac{\zeta}{R_2}\right)\sigma_\zeta\right] + \frac{[\alpha_2(1 + \zeta/R_2)\tau_{1\zeta}]_{,1} + [\alpha_1(1 + \zeta/R_1)\tau_{2\zeta}]_{,2}}{\alpha_1\alpha_2} = 0.
\end{gathered}
\tag{4}
$$

Integration of these gives

$$
\begin{gathered}
\left(1 + \frac{\zeta}{R_1}\right)\left(1 + \frac{\zeta}{R_2}\right)\tau_{i\zeta} = \frac{\tau_{i\zeta m}}{1 + \zeta/R_i}, \\
\left(1 + \frac{\zeta}{R_1}\right)\left(1 + \frac{\zeta}{R_2}\right)\sigma_\zeta = \sigma_{\zeta m} - \frac{\zeta}{\alpha_1\alpha_2}\left[\left(\frac{\alpha_2\tau_{1\zeta m}}{1 + \zeta/R_1}\right)_{,1} + \left(\frac{\alpha_1\tau_{2\zeta m}}{1 + \zeta/R_2}\right)_{,2}\right],
\end{gathered}
\tag{5}
$$

with $\sigma_{\zeta m}$ and $\tau_{i\zeta m}$ being the values of σ_ζ and $\tau_{i\zeta}$ for $\zeta = 0$.

Statics of composite shell. In view of the fact that all face-parallel core-layer stresses are neglected we have as expressions for stress couples M_{ij} and middle surface parallel stress resultants N_{ij},

$$
\begin{aligned}
N_{11} &= \left(1 + \frac{c}{R_2}\right)N_{11u} + \left(1 - \frac{c}{R_2}\right)N_{11l}, \\
M_{11} &= c\left[\left(1 + \frac{c}{R_2}\right)N_{11u} - \left(1 - \frac{c}{R_2}\right)N_{11l}\right].
\end{aligned}
\tag{6}
$$

etc. with evident differences between N_{12} and N_{21} as well as between M_{12} and M_{21}. In the same way we have as expressions for components of external force and moment load intensity

$$
\begin{aligned}
p_n &= \left(1+\frac{c}{R_2}\right)\left(1+\frac{c}{R_1}\right)p_{nu} + \left(1-\frac{c}{R_2}\right)\left(1-\frac{c}{R_1}\right)p_{nl} \\
q &= \left(1+\frac{c}{R_2}\right)\left(1+\frac{c}{R_1}\right)q_u + \left(1-\frac{c}{R_2}\right)\left(1-\frac{c}{R_1}\right)q_l \\
m_n &= c\left[\left(1+\frac{c}{R_2}\right)\left(1+\frac{c}{R_1}\right)p_{nu} - \left(1-\frac{c}{R_2}\right)\left(1-\frac{c}{R_1}\right)p_{nl}\right].
\end{aligned} \tag{7}
$$

Furthermore, a load term of the following form is encountered,

$$
s = \frac{1}{2}\left[\left(1+\frac{c}{R_2}\right)\left(1+\frac{c}{R_1}\right)q_u - \left(1-\frac{c}{R_2}\right)\left(1-\frac{c}{R_1}\right)q_l\right], \tag{8}
$$

with this term representing the average transverse normal stress at any station of the shell, in the event that the loads q_u and q_l alone were responsible for this stress.

The above is complemented by expressions for transverse shear stress resultants Q_i. We find, with the help of eqn (5),

$$
Q_1 = \int_{-c}^{c}\left(1+\frac{\zeta}{R_2}\right)\tau_{1\zeta}\,\mathrm{d}\zeta = \frac{2c\tau_{1\zeta m}}{1-(c/R_1)^2}, \tag{9}
$$

with a corresponding expression for Q_2, and we note that eqn (9) in conjunction with the first relation in (5) makes it possible to express the shear stress values $\tau_{i\zeta u}$ and $\tau_{i\zeta l}$ in terms of Q_i.

Differential equations of equilibrium for the composite shell are now obtained by suitable combination of the above results.

Addition of the two relations in (2) gives as one equation of force equilibrium

$$
\frac{(\alpha_2 N_{11})_{,1} + (\alpha_1 N_{21})_{,2} + \alpha_{1,2}N_{12} - \alpha_{2,1}N_{22}}{\alpha_1\alpha_2} + \frac{Q_1}{R_1} + p_1 = 0. \tag{10}
$$

Subtraction of the same two relations gives as one equation of moment equilibrium

$$
\frac{(\alpha_2 M_{11})_{,1} + (\alpha_1 M_{21})_{,2} + \alpha_{1,2}M_{12} - \alpha_{2,1}M_{22}}{\alpha_1\alpha_2} - Q_1 + m_1 = 0. \tag{11}
$$

Two analogous equations follow upon interchange of subscripts.

Two additional equations are obtained by adding and subtracting, respectively, the two normal component equilibrium relations in (3).

Addition of the equations in (3), and observation of eqns (5)–(7), gives the conventional equation of transverse force equilibrium,

$$
\frac{(\alpha_2 Q_1)_{,1} + (\alpha_1 Q_2)_{,2}}{\alpha_1\alpha_2} - \frac{N_{11}}{R_1} - \frac{N_{22}}{R_2} + q = 0. \tag{12}
$$

A further equation, which is required for the sandwich-type shell, is obtained by subtracting the second relation in (3) *from the first relation. We find, making use of eqns* (5) *and* (9),

$$
\sigma_{\zeta m} + \frac{1}{2c}\left(\frac{M_{11}}{R_1} + \frac{M_{22}}{R_2}\right) + \frac{c}{2\alpha_1\alpha_2}\left[\left(\frac{\alpha_2 Q_1}{R_1}\right)_{,1} + \left(\frac{\alpha_1 Q_2}{R_2}\right)_{,2}\right] - s = 0. \tag{13}
$$

We note that this sixth equilibrium equation for the elements of the composite shell has no *relation to the conventional sixth equilibrium equation for shell elements which expresses the condition of moment equilibrium about the normals to the middle surface.*†

We obtain the middle surface normal moment equilibrium equation, now as a seventh equilibrium equation, together with what amounts to an eighth equation of equilibrium, by using the exact expressions for N_{12}, N_{21}, M_{12}, M_{21}, which correspond to (6), in conjunction with the fact that $N_{12u} = N_{21u}$ and $N_{12l} = N_{21l}$. The resultant relations are

$$N_{12} + \frac{M_{12}}{R_1} = N_{21} + \frac{M_{21}}{R_2}, \quad M_{12} + c^2\frac{N_{12}}{R_1} = M_{21} + c^2\frac{N_{21}}{R_2}. \tag{14}$$

We note that while it is often assumed that the second of these relations is effectively equivalent to a "constitutive" relation $M_{12} = M_{21}$, there is no need and no obvious advantage, in making such an assumption in this place. Beyond this we can say that, while the first relation in (14) is entirely a statement of two-dimensional statics, the second may be thought of as a consequence of a mixture of constitutive and equilibrium considerations, inasmuch as the form of this relation does depend on information on the three-dimensional nature of the state of stress, as previously discussed for the problem of the homogeneous shell[3].

STRESS STRAIN RELATIONS FOR COMPOSITE SHELL

We derive stress strain relations through the use of the theorem of minimum complementary energy, as first employed by Trefftz for homogeneous shells without consideration of the effect of transverse stresses[4].

Designating the complementary energy of face layers and core layers by Π_t and Π_c, respectively, we obtain stress strain relations through the device of extremizing $\Pi_t + \Pi_c$, with the constraint differential equations of equilibrium for the composite shell incorporated into the variational equation through the device of Lagrange multipliers which then may be identified with the appropriate displacement components, as follows.

$$\begin{aligned} 0 = {} & \delta(\Pi_t + \Pi_c) \\ & + \delta \int\int \Bigg\{ \bigg[(\alpha_2 N_{11})_{,1} + (\alpha_1 N_{21})_{,2} + \alpha_{1,2} N_{12} - \alpha_{2,1} N_{22} + \alpha_1\alpha_2 \frac{Q_1}{R_1} \bigg] u_1 \\ & + [(\alpha_2 M_{11})_{,1} + (\alpha_1 M_{21})_{,2} + \alpha_{1,2} M_{12} - \alpha_{2,1} M_{22} - \alpha_1\alpha_2 Q_1]\beta_1 \\ & + [\ldots]u_2 + [\ldots]\beta_2 + \alpha_1\alpha_2 \bigg[N_{12} - N_{21} + \frac{M_{12}}{R_1} - \frac{M_{21}}{R_2} \bigg]\omega \\ & + \alpha_1\alpha_2 \bigg[M_{12} - M_{21} + c^2\bigg(\frac{N_{12}}{R_1} - \frac{N_{21}}{R_2}\bigg)\bigg]\lambda + \bigg[(\alpha_2 Q_1)_{,1} + (\alpha_1 Q_2)_{,2} \\ & - \alpha_1\alpha_2 \bigg(\frac{N_{11}}{R_1} + \frac{N_{22}}{R_2}\bigg)\bigg] w + \bigg[\alpha_1\alpha_2 \sigma_{\zeta m} + \frac{\alpha_1\alpha_2}{2c}\bigg(\frac{M_{11}}{R_1} + \frac{M_{22}}{R_2}\bigg) \\ & + \frac{c}{2}\bigg\langle \bigg(\frac{\alpha_2 Q_1}{R_1}\bigg)_{,1} + \bigg(\frac{\alpha_1 Q_2}{R_2}\bigg)_{,2} \bigg\rangle \bigg] k \Bigg\} \, d\xi_1 \, d\xi_2. \end{aligned} \tag{15}$$

In this, the u_i, w, β_i and ω are readily identified as effective components of translational and rotational displacements. As regards the multiplier k it was noted in [2] that "there is no immediate simple geometrical interpretation, [although] such an interpretation in terms of an average transverse normal strain might be deduced." Similarly, there is no immediate simple geometrical interpretation for the multiplier λ.

It remains to express the complementary energy contributions Π_t and Π_c in terms of stress resultants and couples and in terms of the transverse normal stress measure $\sigma_{\zeta m}$.

†The corresponding relation in [2] is there written without the Q_i-terms which appear above. The reason for this is the "provisional" definition of $\sigma_{\zeta m}$ in eqn (33) which is not consistent with the significance of $\sigma_{\zeta m}$ in eqn (40). However, the considerations which follow show that the Q_i-terms in (13) may in fact be considered negligible.

Assuming the face membranes to be isotropic we have as expression for Π_t,

$$\Pi_t = \int\int \frac{1}{2tE_t}[N_{11u}^2 + N_{22u}^2 - 2\nu N_{11u}N_{22u} + 2(1+\nu)N_{12u}^2]\left(1+\frac{c}{R_1}\right)\left(1+\frac{c}{R_2}\right)\alpha_1\alpha_2\, d\xi_1\, d\xi_2$$
$$+\int\int \frac{1}{2tE_t}[N_{11l}^2 + \cdots]\left(1-\frac{c}{R_1}\right)\left(1-\frac{c}{R_2}\right)\alpha_1\alpha_2\, d\xi_1\, d\xi_2. \tag{16}$$

Equation (16) is transformed into an expression involving stress resultants and couples for the composite shell by writing, on the basis of eqn (6),

$$2\left(1+\frac{c}{R_2}\right)N_{11u} = N_{11} + \frac{M_{11}}{c}, \qquad 2\left(1-\frac{c}{R_2}\right)N_{11l} = N_{11} - \frac{M_{11}}{c}, \text{ etc.} \tag{17}$$

In what follows we restrict attention to cases for which $c/R \ll 1$.† Therewith, and with the definitions,

$$C = 2tE_t, \quad D = c^2C, \tag{18}$$

for stiffness coefficients, we have

$$\Pi_t = \frac{1}{2}\int\int\left\{\frac{1}{C}[N_{11}^2 + N_{22}^2 - 2\nu N_{11}N_{22} + 2(1+\nu)N_{12}N_{21}]\right.$$
$$\left.+\frac{1}{D}[M_{11}^2 + M_{22}^2 - 2\nu M_{11}M_{22} + 2(1+\nu)M_{12}M_{21}]\right\}\alpha_1\alpha_2\, d\xi_1\, d\xi_2. \tag{19}$$

Next, with the face-layer-parallel core stresses σ_1, σ_2, τ_{12} assumed negligible, we have as expression for Π_c,

$$\Pi_c = \int\int\int_{-c}^{c}\left(\frac{\sigma_\zeta^2}{2E_c} + \frac{\tau_{1\zeta}^2 + \tau_{2\zeta}^2}{2G_c}\right)\left(1+\frac{\zeta}{R_1}\right)\left(1+\frac{\zeta}{R_2}\right)d\zeta\,\alpha_1\alpha_2\, d\xi_1\, d\xi_2. \tag{20}$$

In this we now take the stresses in accordance with eqn (5), and we again neglect terms of the order ζ/R in comparison with unity. Therewith and with the help of the defining relation (9), we now have

$$\Pi_c = \int\int\left\{\frac{c}{E_c}\left[\sigma_{\zeta m}^2 + \frac{1}{12}\left(\frac{(\alpha_2Q_1)_{,1} + (\alpha_1Q_2)_{,2}}{\alpha_1\alpha_2}\right)^2\right] + \frac{Q_1^2 + Q_2^2}{4cG_c}\right\}\alpha_1\alpha_2\, d\xi_1\, d\xi_2. \tag{21}$$

In our earlier work[2] eqn (21) had been transformed by elimination of the Q_i-derivative terms through use of the transverse force equilibrium eqn (12), thereby introducing a term $N_{11}/R_1 + N_{22}/R_2 - q$ into Π_c. We now undertake a significant simplification of the results to be obtained by showing that it is, in effect, rational to neglect the Q_i-*derivative* terms in the above in comparison with the Q_i-terms themselves, as long as E_c is of the same order of magnitude as G_c, and as long as it is assumed that significant changes of stress resultants and couples of the composite shell require distances of an order L which are large compared to $2c$, that is, as long as it is required that the solutions to be obtained are such as to justify the use of a two-dimensional theory. For a proof of the correctness of our statement we need only observe that with $Q_{i,j}/\alpha_j = O(Q_i/L)$ the Q_i-derivative terms in (21) are in fact small of relative order $(c/L)^2$ in comparison with the Q_i-terms, and so may rationally be neglected in the expression for Π_c.

Remarkably, it is possible to justify neglect of the Q_i-derivative terms in the last term of the variational eqn (15) in a manner which is in every way consistent with the above. To see this we

†Except for evaluation of the term $N_{12u}^2(1+c/R_1)(1+c/R_2) + N_{12l}^2(1-c/R_1)(1-c/R_2)$ in (16). Remarkably, this term reduces to the form $2N_{12}N_{21} + 2M_{12}M_{21}/c^2$, without any restriction on the magnitude of c/R. Our statement here represents a correction of the corresponding result in[2].

take account of the moment eqn (11) in order to establish that $Q_i = O(M/L)$. From this it follows that the Q_i-derivative terms in the last term in (15) are in fact small of relative order $(c/L)^2$ in comparison with the M_{ii}-terms, consistent with the formulation in[2].

We refrain from rewriting eqn (15) with the appropriate expressions for Π_t and Π_c, and with the two aforementioned simplifications involving neglect of Q_i-derivative terms, and proceed to state the stress strain relations for the composite shell which follow as a consequence of this variational equation

$$
\begin{aligned}
&N_{11} - \nu N_{22} = C\epsilon_{11}, \quad N_{22} - \nu N_{11} = C\epsilon_{22},\\
&N_{12} = \frac{C}{1+\nu}\left(\epsilon_{21} + \omega + c^2\frac{\lambda}{R_2}\right), \qquad N_{21} = \frac{C}{1+\nu}\left(\epsilon_{12} - \omega - c^2\frac{\lambda}{R_1}\right),\\
&M_{11} - \nu M_{22} = D\left(\kappa_{11} - \frac{k}{2cR_1}\right), \qquad M_{22} - \nu M_{11} = D\left(\kappa_{22} - \frac{k}{2cR_2}\right),\\
&M_{12} = \frac{D}{1+\nu}\left(\kappa_{21} + \lambda + \frac{\omega}{R_2}\right), \qquad M_{21} = \frac{D}{1+\nu}\left(\kappa_{12} - \lambda - \frac{\omega}{R_1}\right),\\
&Q_1 = 2cG_c\gamma_1, \quad Q_2 = 2cG_c\gamma_2, \quad 2c\sigma_{\zeta m} = -E_c k.
\end{aligned}
\tag{22}
$$

In these we have

$$
\begin{aligned}
&\epsilon_{11} = \frac{u_{1,1}}{\alpha_1} + \frac{\alpha_{1,2}u_2}{\alpha_1\alpha_2}, \quad \epsilon_{12} = \frac{u_{2,1}}{\alpha_1} - \frac{\alpha_{1,2}u_1}{\alpha_1\alpha_2}, \quad \text{etc.}\\
&\kappa_{11} = \frac{\varphi_{1,2}}{\alpha_1} + \frac{\alpha_{1,2}\varphi_2}{\alpha_1\alpha_2}, \quad \kappa_{12} = \frac{\varphi_{2,1}}{\alpha_1} - \frac{\alpha_{1,2}u_1}{\alpha_1\alpha_2}, \quad \text{etc.}\\
&\gamma_1 = \varphi_1 + \frac{w_{,1}}{\alpha_1} - \frac{u_1}{R_1}, \quad \text{etc.}
\end{aligned}
\tag{23}
$$

While ϵ_{11}, ϵ_{22}, κ_{11}, κ_{22} are the usual expressions for midsurface normal strains and bending strains, the quantities ϵ_{12}, ϵ_{21}, κ_{12}, κ_{21} have no such direct geometrical significance, whereas $\epsilon_{12} + \epsilon_{21}$ and $\kappa_{12} + \kappa_{21}$ are midsurface shearing strain and twisting strain, respectively.

We make two observations in regard to the form of the stress strain relations (22). The first of these is as follows.

We may use the two equilibrium relations (14) in order to eliminate ω and λ from the expressions for N_{12}, N_{21}, M_{12}, M_{21}. When this is done we obtain, except for terms of relative order c^2/R^2,

$$
\omega = \frac{\epsilon_{12} - \epsilon_{21}}{2} - \frac{c^2}{2}\left(\frac{\kappa_{12}}{R_1} - \frac{\kappa_{21}}{R_2}\right), \quad \lambda = \frac{\kappa_{12} - \kappa_{21}}{2} - \frac{1}{2}\left(\frac{\epsilon_{12}}{R_1} - \frac{\epsilon_{21}}{R_2}\right)
\tag{24}
$$

and therewith

$$
\begin{aligned}
&N_{12} = \frac{C}{1+\nu}\left[\frac{\epsilon_{12} + \epsilon_{21}}{2} - \frac{c^2}{2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\kappa_{12}\right],\\
&M_{12} = \frac{D}{1+\nu}\left[\frac{\kappa_{12} + \kappa_{21}}{2} - \frac{1}{2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\epsilon_{12}\right],
\end{aligned}
\tag{25}
$$

with corresponding expressions for N_{21} and M_{21}.†

†We note that in the event that the second relation in (14) is replaced by the statement that $M_{12} = M_{21}$ we shall then have

$$
M_{12} = M_{21} = \frac{1}{2}\frac{D}{1+\nu}\left[\kappa_{12} + \kappa_{21} + \omega\left(\frac{1}{R_2} - \frac{1}{R_1}\right)\right],
$$

and the terms with λ in the expressions for N_{12} and N_{21} will be absent. From this it follows that

$$
(1+\nu)(N_{12} + N_{21}) = C(\epsilon_{12} + \epsilon_{21}),
$$

and furthermore, with $(1+\nu)(N_{12} - N_{21}) = C(\epsilon_{12} - \epsilon_{21} + 2\omega) = (1+\nu)(M_{21}/R_2 - M_{12}/R_1)$ we may eliminate ω from the expressions for $M_{12} = M_{21}$ in such a way that, except for terms of relative order c^2/R^2,

$$
M_{12} = M_{21} = \frac{1}{2}\frac{D}{1+\nu}\left[\kappa_{12} + \kappa_{21} + \left(\frac{1}{R_2} - \frac{1}{R_1}\right)\frac{\epsilon_{12} - \epsilon_{21}}{2}\right].
$$

Our second observation concerns the appearance of the terms with k in the two relations involving M_{11} and M_{22}, and the relation between k and $\sigma_{\zeta m}$ in eqns (22).

We may, in order to eliminate the explicit appearance of the Cosserat-concept from the above, proceed as follows. We combine the last equation in (22) with the simplified version of the Cosserat-type equilibrium eqn (13), in order to obtain the relation $k = (M_{11}/R_1 + M_{22}/R_2)/E_c$. We introduce this result into the equations involving M_{11} and M_{22} in (22) and have therewith as "conventional" stress strain relations

$$M_{11} - \nu M_{22} + \frac{D}{2cR_1E_c}\left(\frac{M_{11}}{R_1} + \frac{M_{22}}{R_2}\right) = D\kappa_{11},$$
$$M_{22} - \nu M_{11} + \frac{D}{2cR_2E_c}\left(\frac{M_{11}}{R_1} + \frac{M_{22}}{R_2}\right) = D\kappa_{22}. \quad (26)$$

It is evident, as first noted in[2], that there will be significant effects of the transverse normal stress deformability of the core layer on the bending stiffness of the composite shell whenever E_c is small enough to result in the order of magnitude relation $(tc/R^2)(E_t/E_c) = O(1)$.

REFERENCES

1. P. M. Naghdi, Direct formulation of some two-dimensional problems of mechanics. *Proc. 7th U.S. National Congr. Appl. Mech.*, 3–21 (1974).
2. E. Reissner, Small bending and stretching of sandwich-type shells. *NACA TN* 1832 (1949). (Also NACA Report 975, 1–26, 1950).
3. E. Reissner, One some problems in shell theory. *Proc. 3rd Symp. Naval Struct. Mech.* 87–94, Pergamon Press, Oxford (1960).
4. E. Trefftz, Ableitung der Schalenbiegungsgleichungen mit dem Castigliano'schem Prinzip. *Z. f. ang. Math. & Mech.* **15**, 101–108 (1935).